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1998 J. Phys. A: Math. Gen. 31 209

(http://iopscience.iop.org/0305-4470/31/1/021)

J. Phys. A: Math. Gen. **31** (1998) 209–229. Printed in the UK PII: S0305-4470(98)86207-0

Kink manifolds in $(1+1)$-dimensional scalar field theory

A Alonso Izquierdo†, M A González León† and J Mateos Guilarte‡

† Departamento de Estadística y Matemática Aplicadas, Facultad de Ciencias, Universidad de Salamanca, Salamanca 37008, Spain

‡ Departamento de Física, Facultad de Ciencias, Universidad de Salamanca, Salamanca 37008, Spain

Received 22 July 1997

Abstract. The general structure of kink manifolds in $(1+1)$-dimensional complex scalar field theory is described by analysing three special models. New solitary waves are reported. Kink energy sum rules arise between different types of solitary waves.

1. Introduction

Kinks are topological defects arising in several domains of physics. They exists in one-dimensional condensed matter systems, polyacetilen being an example [1], and induce exotic phenomena such as electric charge fractionization. In hydrodynamics [2], kinks are solitary waves of a special kind, for example shockwaves in fluids. In a $(3+1)$-dimensional spacetime, kinks are non-dispersive solutions of the field equations independent of two of the three spatial coordinates; in this disguise of domain walls, kinks also play a very important role in cosmology [3].

In this paper we shall focus on the theoretical study of kinks in relativistic $(1+1)$-dimensional scalar field theory. The systems we shall consider thus have an associated energy-momentum tensor and we shall stick to the definition of solitary waves given in [4]:

A solitary wave is a non-singular solution of the nonlinear coupled field equations of finite energy such that their energy density has a spacetime dependence of the form

$$\varepsilon(\boldsymbol{x}, t) = \varepsilon(\boldsymbol{x} - \boldsymbol{v}t)$$

where $\boldsymbol{v}$ is some velocity vector.

In models with a single real scalar field in $(1+1)$ dimensions, much is known about these peculiar lumps both at the classical and quantum levels. When the field theory involved encompasses several scalar fields, however, the search for kinks and the study of their properties become much more problematic. To undertake such an enterprise is not merely an academic problem: the order parameters distinguishing the different phases of liquid crystals, [5], are scalar fields in vector—or tensor—irreducible representations of the $O(N)$ group. Thus, kink solutions correspond to one-dimensional topological defects present in this kind of physical system.

Rajaraman [4] envisaged a trial orbit method to find new kinds of kinks in the so-called MSTB model [6], having no analogues in one-component scalar field theory. The model describes the dynamics of a complex scalar field by means of an action which is a deformation of the O(2)-linear sigma model. Instead of spontaneous symmetry breaking of O(2) by a degenerated S^1 vacuum manifold the O(2) symmetry group is explicitly broken to $\mathbb{Z}_2$ by a mass term; only invariance under $\phi \to -\phi$ survives. From the point of view of quantum field theory, this deformation is very natural because in $(1+1)$ dimensions infrared divergences forbid the existence of Goldstone bosons, according to a theorem of Coleman [7]. Even if it is absent in the classical action, a mass term will be generated by quantum corrections.

For time-independent configurations the field equations become ordinary differential equations. The search for kinks is therefore tantamount to a dynamical system of Lagrangian type where the energy of the static configurations of the field theory becomes the action. Rajaraman realized that kinks correspond to very special trajectories in the associated N-dimensional mechanical problem, where N in the number of scalar fields. Separatrices between bounded and unbounded motion in the mechanical system are in one-to-one correspondence with kinks in the scalar field theory. The trial orbit method is a procedure used to elucidate whether a given trajectory is a separatrix but does not provide a general scheme to find all the kinks. Moreover, when $N > 2$ the method becomes very cumbersome.

The total manifold of kinks in the MSTB model was described for the first time by Magyari and Thomas [8], who noticed that the dynamical system was indeed Hamiltonian and integrable. They used the method of commuting Hamiltonian flows as proposed by Hieratinta and Fordy [9] to show that, at least implicitly, all the trajectories are known and one needs only to impose asymptotic conditions in an appropriate way to obtain the separatrices. In a seminal paper, Ito [10] gave a complete analytical treatment of the kink manifold by observing that the Hamiltonian system describing kinks as special trajectories is separable into elliptic coordinates and applying the Hamilton–Jacobi method. In particular, a very awkward kink energy sum rule, established previously as a quasi-empirical fact, was fully explained.

The aim of this paper is to show that the MSTB model, enjoying such interesting physical and mathematical features, is not unique. Here we shall propose and study two models that we call, respectively, A and B to find that they share all the peculiarities of the MSTB model. Model A is a deformation of the Chern–Simons linear sigma model where the potential energy density is of the form appearing in Chern–Simons–Higgs planar gauge systems [11]. As in the MSTB model there is a deformation parameter varying in a finite range such that a generalized conserved charge, which tends to the SO(2)-isospin generator when the parameter goes to zero, still exists. Model B is a different deformation of the O(2)-linear sigma model from that leading to the MSTB model. A conserved charge also arises but only for a single value of the deformation parameter.

The unifying property of all these three models is that the dynamical systems associated with the search for kinks are completely integrable. They are separable by using either elliptic coordinates—models MSTB and A—or parabolic coordinates—model B. The dynamical systems are thus Liouville integrable mechanical models of type I (models MSTB and A) and type III (model B) see [12]. Automatically, kink energy sum rules occur as a characteristic of these very special field theories.

Following the present trend of studying manifolds of quantum field we shall analyse how general structures are shared by the different models. The renormalization group flow in the parameter space meets these special models when a hidden, nonlinear, 'deformed' isospin charge is conserved even though the O(2)-symmetry is explicitly broken. Spontaneous symmetry breakdown of continuous groups is forbidden by Coleman's theorem because Goldstone bosons cannot exist. Models A and B, as well as the MSTB model, are not invariant with respect to the O(2) group and do not have a continuous vacuum manifold as requested by Coleman's theorem in $(1+1)$-dimensional field theory. However, they do retain some invariance properties of a new kind in field theory. There are strong analogies with the Zamolodchikov c-theorem [13]: deformations in conformal field theory models leading to integrable models are the most interesting ones. Despite the appearance of massive particles breaking the conformal symmetry the 'special' systems are solvable due to the existence of a hidden infinite-dimensional symmetry group.

This paper is organized as follows. In section 2 we introduce models A and B, discuss their configuration spaces and particle spectra and recall the kink manifold of the MSTB model. Section 3 is devoted to the analysis of the kink manifold of models A and B and the statement of the new kink energy sum rules. Finally, in section 4 we briefly comment on several extensions of this work.

2. Models A and B versus the MSTB model

We shall look at $(1+1)$-dimensional complex scalar field theory models with dynamics governed by the action

$$S = \int \mathrm{d}^2 y \,\{\tfrac{1}{2}\partial_\mu \chi^* \partial^\mu \chi - \bar{V}(\chi^*, \chi)\}$$

here, $\chi(y_\mu) = \chi_1(y_\mu) + i\chi_2(y_\mu)$ is a complex scalar field and we choose the metric $g_{\mu\nu}$, $\mu, \nu = 0, 1$ to be of the form $g_{00} = -g_{11} = 1$, $g_{12} = g_{21} = 0$ in two-dimensional Minkowski space. Models A and B differ in their potential energy densities:

model A $$\bar{V}_{\mathrm{A}} = \frac{\lambda^4}{4m^2}\chi^*\chi\left(\chi^*\chi - \frac{m^2}{\lambda^2}\right)^2 + \frac{\beta^2}{2}\chi_2^2\left(\chi^*\chi - \frac{m^2}{\lambda^2}\left(1 - \frac{\beta^2}{2\lambda^2}\right)\right)$$

model B $$\bar{V}_{\mathrm{B}} = \frac{\lambda^2}{2}\left(4\chi_1^2 + \chi_2^2 - \frac{m^2}{\lambda^2}\right)^2 + 2\lambda^2\chi_1^2\chi_2^2.$$

This is to be compared with the MSTB model where the choice of potential energy density is:

$$\bar{V}_{\mathrm{MSTB}} = \frac{\lambda^2}{4}\left(\chi^*\chi - \frac{m^2}{\lambda^2}\right)^2 + \frac{\beta^2}{4}\chi_2^2.$$

The λ, m and β coupling constants are of inverse length dimension.

We study the range $\beta^2 < \lambda^2$ in parameter space because in this regime the structure of the kink manifold is richer. Introducing non-dimensional variables $\chi \to \frac{m}{\lambda}\phi$, $y_\mu \to \frac{\sqrt{2}}{m}x_\mu$, $\frac{\beta^2}{\lambda^2} \to \sigma^2$ and $\frac{\beta^2}{m^2} \to \gamma^2$, we find simpler expressions for the action:

$$\begin{aligned}
S &= \frac{m^2}{\lambda^2}\int \mathrm{d}^2 x \,\left\{\frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - V(\phi^*, \phi)\right\} \\
V_{\mathrm{A}} &= \frac{\phi^*\phi}{2}(\phi^*\phi - 1)^2 + \sigma^2\phi_2^2\left(\phi^*\phi - 1 + \frac{\sigma^2}{2}\right) \\
V_{\mathrm{B}} &= (4\phi_1^2 + \phi_2^2 - 1)^2 + 4\phi_1^2\phi_2^2 \\
V_{\mathrm{MSTB}} &= \frac{1}{2}(\phi^*\phi - 1)^2 + \frac{\gamma^2}{2}\phi_2^2.
\end{aligned} \tag{1}$$

2.1. *Configuration space and particle spectra*

For the set of time-independent configurations $\phi(x^0, x^1) = f(x^1)$ the energy functional is:

$$E = \frac{m^3}{\sqrt{2}\lambda^2}\int \mathrm{d}x^1 \,\left\{\frac{1}{2}\frac{\mathrm{d}f^*}{\mathrm{d}x^1}\cdot\frac{\mathrm{d}f}{\mathrm{d}x^1} + V(f^*, f)\right\}. \tag{2}$$

The configuration space is the set of maps $f : \mathbb{R} \to \mathbb{C}$ such that $E < +\infty$: $\mathcal{C} = \{f(x^1)/E < +\infty\}$. This requires f to be continuous and satisfy the asymptotic conditions:

$$\lim_{x^1 \to \pm\infty} \frac{\mathrm{d}f}{\mathrm{d}x^1} = 0 \qquad \lim_{x^1 \to \pm\infty} f(x^1) = v \tag{3}$$

where v is a constant belonging to $\mathcal{M}$, the set of zeros of V. Note that in our models V is always semidefinite positive.

In all three cases, the set $\mathcal{M}$ is discrete:

$$\mathcal{M}_{\mathrm{MSTB}} \equiv \mathbb{Z}_2 = \{v^{(\pm)}_{\mathrm{MSTB}} = \pm 1\}$$
$$\mathcal{M}_{\mathrm{A}} \equiv \mathbb{Z}_2 \sqcup \mathbb{Z}_2 \sqcup e = \{v^{(\pm 1)}_{\mathrm{A}} = \pm 1, v^{(\pm i)}_{\mathrm{A}} = \pm \mathrm{i}\bar{\sigma}, v^{(0)}_{\mathrm{A}} = 0\}$$
$$\mathcal{M}_{\mathrm{B}} \equiv \mathbb{Z}_2 \sqcup \mathbb{Z}_2 = \{v^{(\pm 1)}_{\mathrm{B}} = \pm \tfrac{1}{2}, v^{(\pm i)}_{\mathrm{B}} = \pm \mathrm{i}\}$$

where $\bar{\sigma} = \sqrt{1 - \sigma^2}$. We refer to $\mathcal{M}$ as the vacuum manifold because in the quantum version of the theory points in $\mathcal{M}$ are the expectation values of the quantum field operator at the ground states or vacua of the system. The vacuum degeneracy—i.e. the existence of more than one element in $\mathcal{M}$—is related to the symmetry breaking. In our models, besides two-dimensional Poincaré invariance, there is a global or internal symmetry with respect to the discrete group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $\phi_1 \to -\phi_1$ and $\phi_2 \to -\phi_2$. The vacuum manifold is in general the orbit of one element by the group action. In our model, however, $\mathcal{M}$ is the union of several orbits:

$$\mathcal{M}_{\mathrm{MSTB}} = G/H_{v^{(\pm)}} = \mathbb{Z}_2$$
$$\mathcal{M}_{\mathrm{A}} = G/H_{v^{(\pm 1)}_{\mathrm{A}}} \sqcup G/H_{v^{(\pm i)}_{\mathrm{A}}} \sqcup G/H_{v^{(0)}_{\mathrm{A}}} = \mathbb{Z}_2 \sqcup \mathbb{Z}_2 \sqcup e$$
$$\mathcal{M}_{\mathrm{B}} = G/H_{v^{(\pm 1)}_{\mathrm{B}}} \sqcup G/H_{v^{(\pm i)}_{\mathrm{B}}} = \mathbb{Z}_2 \sqcup \mathbb{Z}_2$$

where H_v is the little group of the vacuum v and it is the surviving symmetry group when quantizing around v. Unlike the MSTB model, the moduli space of vacua $\mathcal{M}^0 = \mathcal{M}/G$ has more than one element in models A and B:

$$\mathcal{M}^0_{\mathrm{A}} = \mathcal{M}_{\mathrm{A}}/G = e \sqcup e \sqcup e \qquad \mathcal{M}^0_{\mathrm{B}} = \mathcal{M}_{\mathrm{B}}/G = e \sqcup e.$$

The zeros of V are also critical points in our models satisfying:

$$\left.\frac{\partial V}{\partial \phi_a}\right|_{\phi = v} = 0.$$

Therefore, they are constant solutions of the field equations

$$\Box \phi_a = -\frac{\partial V}{\partial \phi_a} \qquad a = 1, 2. \tag{4}$$

Small deformations of $\phi^0(x) = v$ that are still stable solutions of (4) corresponds in the partner quantum theory to the fundamental quanta. The pattern of symmetry breaking arises as follows

$$\phi_v(\kappa; x) = v + \sum_\kappa a(\kappa) \mathrm{e}^{\mathrm{i}\kappa x} \qquad \kappa = (\kappa^0, \kappa^1) \tag{5}$$

is a plane wave solution of (4) if the dispersion relation

$$\delta_{ab}(\kappa^0)^2 = (\kappa^1)^2 \delta_{ab} + M^2_{ab}(v) \qquad M^2_{ab}(v) = \frac{\partial^2 V}{\partial \phi_a \partial \phi_b}(v)$$

holds. $M^2_{ab}(v)$ is therefore the mass matrix at the chosen critical point and the symmetry of the 'particle' spectrum is precisely H_v. Note that all $v \in \mathcal{M}$ under discussion are minima of

V; hence there are no negative eigenvalues of $M^2_{ab}(v)$ and the small deformation solutions (5) around the constant minima v depend on time as: $e^{i\kappa^0 x^0}$, i.e. do not increase without bounds in time.

Understanding our models as physically describing the continuum approximation to a one-dimensional crystal with a two-component order parameter the 'particle' spectra are as follows.

(1) There is one phase with two phonon branches in the MSTB model:

$$M^2(v^{(\pm)}_{\rm MSTB}) = \begin{pmatrix} 4 & 0 \\ 0 & \gamma^2 \end{pmatrix} \frac{m^2}{2}.$$

(2) There are three phases with two phonon branches per phase in model A:

$$M^2(v^{(\pm 1)}_{\rm A}) = \begin{pmatrix} 4 & 0 \\ 0 & \sigma^4 \end{pmatrix} m^2 \qquad M^2(v^{(\pm i)}_{\rm A}) = \begin{pmatrix} \sigma^4 & 0 \\ 0 & 4\bar{\sigma}^4 \end{pmatrix} m^2$$

$$M^2(v^{(0)}_{\rm A}) = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\sigma}^4 \end{pmatrix} m^2.$$

(3) Two phases with two phonon branches per phase form the particle spectrum of model B:

$$M^2(v^{(\pm 1)}_{\rm B}) = \begin{pmatrix} 32 & 0 \\ 0 & 2 \end{pmatrix} m^2 \qquad M^2(v^{(\pm i)}_{\rm B}) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} m^2.$$

The symmetry group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is broken by the choice of the $v^{(\pm)}$ vacuum to the $H \equiv \mathbb{Z}_2$ subgroup: $\phi_2 \to -\phi_2$ in the MSTB model. The two phonon branches have different masses or 'energy gaps', even at the continuous symmetry $\gamma^2 = 0$ limit. In model A there are three cases: (a) when the vacuum is $v^{(0)}_{\rm A}$, the symmetry under $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is unbroken: there is no degeneracy in the particle spectrum because the internal O(2) symmetry is explicitly broken to $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ by the $\sigma^2\phi_2^2(\phi^*\phi - 1 + \frac{\sigma^2}{2})$ term in the Lagrangian. (b) The choice of $v^{(\pm 1)}_{\rm A}$ spontaneously breaks the $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry to $H_1 \equiv \mathbb{Z}_2 : \phi_2 \to -\phi_2$. (c) If we choose $v^{(\pm i)}_{\rm A}$ as the little group and the unbroken symmetry is $H_i \equiv \mathbb{Z}_2 : \phi_1 \to -\phi_1$. In model B one can choose either, (a) $v^{(\pm 1)}_{\rm B}$ with little group H_1, or, (b) $v^{(\pm i)}_{\rm B}$, little group H_i, as the vacuum. The degeneracy of the spectrum on the vacua $v^{(\pm i)}_{\rm B}$ is accidental: it is due to the very special behaviour of the potential density energy around those points.

We can look at the MSTB and B models as members of the family characterized by the potential energy densities:

$$V(\phi^*, \phi) = \frac{1}{2}(\alpha_1\phi_1^2 + \alpha_2\phi_2^2 - 1)^2 + \frac{\beta_1^2}{2}\phi_1^2 + \frac{\beta_2^2}{2}\phi_2^2 + \frac{\gamma_1^2}{4}\phi_1^4 + \frac{\gamma_{12}^2}{2}\phi_1^2\phi_2^2 + \frac{\gamma_2^2}{4}\phi_2^4$$

where $\alpha_1, \alpha_2, \beta_1^2, \beta_2^2, \gamma_1^2, \gamma_{12}^2$ and γ_2^2 are 'bare' non-dimensional parameters. Ultraviolet divergences are controlled by normal ordering in the quantum theory, but the need arises to introduce a renormalization 'point' μ^2 and the dependence of the renormalized parameters on μ^2 is determined by the renormalization group equation. One special solution, a specific renormalization group flow, might lead to the 'point':

$$\alpha_1^R(\mu^2) = \alpha_2^R(\mu^2) = 1 \qquad (\beta_1^R)^2 = (\beta_2^R)^2 = (\gamma_1^R)^2 = (\gamma_{12}^R)^2 = (\gamma_2^R)^2|_{\mu^2} = 0$$

in the space of quantum field theory models in the family. This point corresponds to the linear O(2)-sigma model which enjoys a continuous symmetry group $G = {\rm O}(2)$. The vacuum orbit is, however, S^1, and this means that there is no unbroken symmetry. One checks, looking at the particle spectrum of the MSTB model at the limit $\gamma^2 = 0$, that there is a massless particle, a Goldstone boson, as expected when there is spontaneous symmetry breaking of continuous transformations.

Coleman [7] established that in (1 + 1) dimensions the infrared asymptotics of the two-point Green functions of a quantum scalar field forbids poles at $\kappa^2 = 0$; there are no Goldstone bosons in (1+1) dimensions. It is thus impossible to reach the O(2)-sigma model in the renormalization group flow. The closest point to this is the model characterized by:

$$\alpha_1^R(\mu^2) = \alpha_2^R(\mu^2) = 1, (\beta_2^R)^2(\mu^2) = \gamma^2; \beta_1^R(\mu^2) = \gamma_1^R(\mu^2) = \gamma_{12}^R(\mu^2) = \gamma_2^R(\mu^2) = 0$$

i.e. the MSTB model. Even though the ugly term $\frac{\gamma^2}{2}\phi_2^2$ would not be present at the classical level, quantum fluctuations would generate it. In the next section we shall see that this point where the other β- and γ-renormalized parameters are zero is a very particular one; in the $0 < \gamma^2 < 1$ range there is a very rich manifold of kinks. Model B corresponds to another interesting point with similar features,

$$\alpha_1^R(\mu^2) = 4\sqrt{2} \qquad \alpha_2^R(\mu^2) = \sqrt{2} \qquad (\gamma_{12}^R)^2(\mu^2) = 8$$
$$\beta_1^R(\mu^2) = \beta_2^R(\mu^2) = \gamma_1^R(\mu^2) = \gamma_2^R(\mu^2) = 0$$

although in this case it is really a point, not an interval as in the MSTB model.

A completely analogous analysis informs us that model A is a deformation of the Chern–Simons O(2)-sigma model characterized by:

$$V(\phi^*, \phi) = \frac{\phi^*\phi}{2}(\phi^*\phi - 1)^2.$$

This is the $\sigma^2 = 0$ limit of model A with a O(2) continuous symmetry group and a vacuum manifold $\mathcal{M} = S^1 \sqcup \{\text{point}\}$. In the non-symmetric phase there are Goldstone bosons, look at the particle spectrum on $v_A^{(\pm 1)}$ and $v_A^{(\pm i)}$, which cannot exist in quantum theory and the renormalization group flow leads to model A. As in the MSTB model there is a range, $0 < \sigma^2 < 1$, where we shall find an interesting manifold of kinks.

2.2. Configuration space topology and kinks

The configuration spaces of our models are the union of topologically disconnected sectors, see [14]; $\mathcal{C} = \bigsqcup_{\alpha,\beta=1}^{N} \mathcal{C}^{\alpha\beta}$, where N is the order of $\mathcal{M}$. This comes from the identity $\pi_0(\mathcal{C}) = \pi_0(\mathcal{M})$ between the zeroth-order homotopy groups of $\mathcal{C}$ and $\mathcal{M}$, and, in turn, is due to the asymptotic conditions (3). We find in our models: $\pi_0(\mathcal{C}_{\text{MSTB}}) = \mathbb{Z}_2$, $\pi_0(\mathcal{C}_A) = \mathbb{Z}_5$ and $\pi_0(\mathcal{C}_B) = \mathbb{Z}_4$. Accordingly, there exist topological charges,

$$Q_a^{\text{T}} = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{d}\hat{\phi}_a}{\mathrm{d}x} = \hat{\phi}_a(\infty) - \hat{\phi}_a(-\infty)$$

labelling the homotopy classes; here, $\hat{\phi}_a = \frac{\phi_a}{|v_a|}$ and $|v_a|$ are normalization constants given by the a component of the $v^{(\alpha)}$ vacuum, see below. Therefore, $\mathcal{C}_{\text{MSTB}} = \bigsqcup_{\alpha,\beta=1}^{2} \mathcal{C}_{\text{MSTB}}^{\alpha\beta}$, $\mathcal{C}_A = \bigsqcup_{\alpha,\beta=1}^{5} \mathcal{C}_A^{\alpha\beta}$ and $\mathcal{C}_B = \bigsqcup_{\alpha,\beta=1}^{4} \mathcal{C}_B^{\alpha\beta}$ if α and β tell us the vacua reached at $\pm\infty$: $v^{(\alpha)} = \phi(-\infty)$ and $v^{(\beta)} = \phi(\infty)$.

The splitting of the configuration space into four disconnected sectors is not the only non-trivial topological feature of the MSTB model. The very rich topological structure of $\mathcal{C}_{\text{MSTB}}$ was thoroughly analysed [14, 16] in a series of papers. Morse theory of $\mathcal{C}_{\text{MSTB}}$ revealed itself as a powerful tool in the study of both the classical and quantum dynamics of the MSTB model. We shall postpone a parallel study in models A and B to future research. Instead, we shall now focus on searching for time-independent finite-energy solutions of the field equations which are not spatially homogeneous. Some of them live in sectors with non-zero topological charges and are therefore unable to evolve in time to vacuum solutions enjoying topological stability. Others belong to vacuum sectors with a more obscure origin from the topological point of view.

Besides complying with (3), solitary waves satisfy the system of ordinary differential equations

$$\frac{\mathrm{d}^2 f_a}{\mathrm{d}(x^1)^2} = \frac{\partial V}{\partial f_a} \tag{6}$$

where $\phi_a(x^0, x^1) = f_a(x^1)$. To solve system (6) together with the boundary conditions (3) is tantamount to finding the solutions of the Lagrangian dynamical system in which $x^1 = \tau$ plays the role of time, the 'particle' position is determined by $f_a(\tau)$, and the potential energy of the particle is $U(f_a) = -V(f_a)$.

The static field energy is from this perspective the particle action

$$J = \int \mathrm{d}\tau \left\{ \frac{1}{2} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}f_a}{\mathrm{d}\tau} - U(f_a) \right\}$$

and trajectories of finite action, J, in the mechanical problem are in one-to-one correspondence with solitary waves, kinks, of energy $E = J$. Of course, the mechanical analogy is very helpful when one is dealing with a real scalar field because, then, the mechanical system always has a first integral, which is all that we need to find all the solutions. Vector scalar fields of N components lead to N-dimensional dynamical systems which are seldom solvable. Magyari and Thomas [8] realized that the two-dimensional dynamical system arising in connection with the MSTB model is a completely integrable one in the Liouville sense. Moreover, Ito [12] has shown that the mechanical system is Hamilton–Jacobi separable, finding all the trajectories, and hence all the kinks of the MSTB model.

In this case,

$$J = \int \mathrm{d}\tau \left\{ \frac{1}{2} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}f_a}{\mathrm{d}\tau} + \frac{1}{2}(f_a f_a - 1)^2 + \frac{\gamma^2}{2} f_2^2 \right\}$$

and the motion equations of the particle are:

$$\frac{\mathrm{d}^2 f_a}{\mathrm{d}\tau^2} = 2 f_a (f_b f_b - 1) + \delta_{a2} \gamma^2 f_2 \tag{7}$$

to be solved together with the asymptotic conditions

$$\lim_{\tau \to \pm\infty} f_a(\tau) = \pm\delta_{a1} \qquad \lim_{\tau \to \pm\infty} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} = 0. \tag{8}$$

It is convenient to pass to Hamiltonian formalism. The canonical momenta $p_a(\tau) = \frac{\delta J}{\delta f_a}(\tau) = \frac{\mathrm{d}f_a}{\mathrm{d}\tau}$ together with $f_a(\tau)$ are local coordinates in phase space. We must bear in mind that $p_a(\tau) \equiv \frac{\mathrm{d}\phi_a}{\mathrm{d}x^1}$ when going back to the field theory. The mechanical Hamiltonian

$$I_1 = \frac{1}{2} p_a p_a - \frac{1}{2}(f_a f_a - 1)^2 - \frac{\gamma^2}{2} f_2^2$$

leads to the canonical equations, equivalent to (7),

$$\frac{\mathrm{d}f_a}{\mathrm{d}\tau} = \{I_1, f_a\} \qquad \frac{\mathrm{d}p_a}{\mathrm{d}\tau} = \{I_1, p_a\}$$

where the Poisson bracket is defined in the usual way

$$\{F, G\} = \sum_{a=1}^{2} \left(\frac{\partial F}{\partial f_a} \cdot \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a} \cdot \frac{\partial G}{\partial f_a} \right)$$

for any two functionals $F(f_a, p_a)$, $G(f_a, p_a)$ in phase space.

Obviously $\frac{dI_1}{d\tau} = 0$, but our dynamical system has a second invariant

$$I_2 = (f_1 p_2 - f_2 p_1)^2 + \frac{\gamma^2}{2}\left\{p_1^2 - p_2^2 - (f_1^2 - 1)^2 + f_2^2\left(f_2^2 - 2\left(1 - \frac{\gamma^2}{2}\right)\right)\right\} \tag{9}$$

$\frac{dI_2}{d\tau} = 0$, which is in involution with the Hamiltonian: $\{I_1, I_2\} = 0$. According to Liouville's theorem, the two-dimensional mechanical system is completely integrable, all the trajectories can be found, and those with asymptotic behaviour given by (8) will correspond to the solitary waves of the parent scalar field theory.

At this point we pause to explain the singularity of the MSTB model. The $\frac{\gamma^2}{2}\phi_2^2$ term explicitly breaks the O(2)-symmetry of the linear sigma model, the $\gamma^2 = 0$ limit. In the mechanical system the O(2) 'internal' transformations become spatial rotations. Therefore, when $\gamma^2 \neq 0$, the angular momentum,

$$j_{12} = f_1 \frac{df_2}{d\tau} - f_2 \frac{df_1}{d\tau}$$

conserved at the limit $\gamma^2 = 0$, is no longer 'time' independent. There is, however, a second invariant, $I_2 = j_{12}^2 + \gamma F(f_a, p_a)$ becoming the square of j_{12} for $\gamma^2 = 0$, and the deformation meant by the MSTB model is special because it still retains enough symmetry to solve the mechanical system. There is no Lie algebra associated with I_2, because since the invariant is quadratic in f_a, p_a the action on the phase space, given by $\{I_2, f_a\}$ and $\{I_2, p_a\}$, is nonlinear.

From the point of view of (1+1)-dimensional field theory, the energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \cdot \partial^\nu \phi_a - g^{\mu\nu}\mathcal{L}$$

is divergenceless,

$$\partial_\mu T^{\mu\nu} = 0$$

irrespective of the non-zero value of γ^2. $P^\mu = \int dx^1\, T^{0\mu}$ are thus conserved quantities. The O(2) 'isospin' current

$$J^\mu = \varepsilon_{ab}\phi_a \partial^\mu \phi_b$$

is only divergenceless at the limit $\gamma^2 = 0$ and $Q = \int dx^1\, J^0$ is not conserved for non-zero γ^2. At the static limit, we have $T^{00} = E = J$, $T^{10} = T^{01} = 0$, $T^{11} = I_1$, $J^0 = 0$ and $J^1 = j_{12}$.

Defining a deformed isospin current such that

$$J_\gamma^1 J_\gamma^1 = I_2 \qquad J_\gamma^0 J_\gamma^0 = L_2$$

where L_2 is

$$L_2 = \frac{1}{2}J^0 J^0 + \frac{\gamma^2}{2}\left\{\left(\frac{\partial\phi_1}{\partial x^0}\right)^2 - \left(\frac{\partial\phi_2}{\partial x^0}\right)^2 - (\phi_1^2 - 1)^2 + \phi_2^2\left(\phi_2^2 - 2\left(1 - \frac{\gamma^2}{2}\right)\right)\right\}$$

we expect some equation

$$F\left(\frac{\partial L_2}{\partial x^0}, \frac{\partial I_2}{\partial x^1}\right) = 0$$

of a nonlinear character going to $\frac{\partial J^0}{\partial x^0} = \frac{\partial J^1}{\partial x^1}$ when γ^2 goes to 0. We find a very similar situation to that occurring between conformal field theories and models with infinite-dimensional algebra as in $(1+1)$-dimensional Toda field theories and Toda affine models

[15]. There are two differences: (1) the conformal group is infinite dimensional in $(1+1)$ dimensions. We have only one finite-dimensional group O(2) so that we can only solve the static limit of the field theory model. (2) Due to the nonlinear character of the deformation of the O(2) Lie generator we do not have even a finite-dimensional algebra.

Returning to the mechanical system, it is known, see [12], that it is not only integrable but Hamilton–Jacobi separable. Introducing 'elliptic' coordinates,

$$f_1(\tau) = \frac{1}{\gamma} u(\tau) v(\tau) \qquad f_2(\tau) = \frac{1}{\gamma}\sqrt{(u^2(\tau)-\gamma^2)(\gamma^2 - v^2(\tau))}$$
$$v \in [-\gamma, \gamma] \qquad u \in [\gamma, +\infty)$$

the 'action' J is

$$J = \int \mathrm{d}\tau \left\{ \frac{1}{2}(u^2 - v^2)\left[\frac{1}{u^2-\gamma^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 + \frac{1}{\gamma^2 - v^2}\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^2\right] \right.$$
$$\left. + \frac{1}{2(u^2-v^2)}[(u^2-1)^2(u^2-\gamma^2) + (1-v^2)^2(\gamma^2 - v^2)]\right\}. \qquad (10)$$

Formula (10) informs us that the 'action' J depends on u and v in a completely separated form.

This point is of better use in the Hamiltonian formalism. The generalized momenta

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{u^2 - v^2}{u^2 - \gamma^2}\dot{u} \qquad p_v = \frac{\partial L}{\partial \dot{v}} = \frac{u^2 - v^2}{\gamma^2 - v^2}\dot{v}$$

lead to the Hamiltonian,

$$h = \frac{1}{u^2 - v^2}(h_u + h_v)$$

where

$$h_u = \tfrac{1}{2}(u^2 - \gamma^2)p_u^2 - \tfrac{1}{2}(u^2-1)^2(u^2-\gamma^2)$$
$$h_v = \tfrac{1}{2}(\gamma^2 - v^2)p_v^2 - \tfrac{1}{2}(1-v^2)^2(\gamma^2 - v^2).$$

In the phase space submanifold determined by $h = i_1$, where i_1 is a constant of motion, we have $h_u - i_1 u^2 = -h_v - i_1 v^2 = i_2$, with i_2 another constant of motion. It is clear from the analysis that the two invariants, I_1 and I_2, are related to h and h_u. Obviously, $i_1 = I_1$ while

$$i_2 = -I_2 - \gamma^2 I_1.$$

The Hamilton–Jacobi equation

$$\frac{\partial J}{\partial \tau} + \mathcal{H}\left(\frac{\partial J}{\partial u}, \frac{\partial J}{\partial v}, u, v\right) = 0$$

is completely separable

$$\frac{1}{2}(u^2 - \gamma^2)\left(\frac{\mathrm{d}J_u}{\mathrm{d}u}\right)^2 - \frac{1}{2}(u^2-1)^2(u^2-\gamma^2) - i_1 u^2 = i_2$$
$$\frac{1}{2}(\gamma^2 - v^2)\left(\frac{\mathrm{d}J_v}{\mathrm{d}v}\right)^2 - \frac{1}{2}(1-v^2)^2(\gamma^2-v^2) + i_1 v^2 = -i_2 \qquad (11)$$

by writing $J = J_u(u) + J_v(v) - i_1\tau$. The solution of (11)

$$J_u = \mathrm{sign}\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)\int \mathrm{d}u \sqrt{\frac{2i_2 + 2i_1 u^2}{u^2 - \gamma^2} + (u^2-1)^2}$$
$$J_v = \mathrm{sign}\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)\int \mathrm{d}v \sqrt{\frac{-2i_2 - 2i_1 v^2}{\gamma^2 - v^2} + (1-v^2)^2}$$

gives the action of the trajectories. The Hamilton–Jacobi principle also offers the equation satisfied by the trajectories developed by the dynamical system: if $\beta_1 \in \mathbb{R}$ is a constant, $\frac{\partial J}{\partial i_2} = \beta_1$ means that

$$\operatorname{sign}\dot{u}\int\frac{\mathrm{d}u}{\sqrt{\frac{2i_2+2i_1u^2}{u^2-\gamma^2}+(1-u^2)^2(u^2-\gamma^2)}} = \beta_1$$
$$+\operatorname{sign}\dot{v}\int\frac{\mathrm{d}v}{\sqrt{\frac{-2i_2-2i_1v^2}{\gamma^2-v^2}+(1-v^2)^2(\gamma^2-v^2)}}. \tag{12}$$

We do not attempt to integrate (12) in the general case but focus on the values $i_1 = i_2 = 0$ which correspond to finite J action. These special trajectories, complying with (8), are thus the kinks of the original field theory having finite energy $E = J$. The kink manifold of the MSTB model is in one-to-one correspondence with the solutions of the equations

$$\left|\frac{(u+1)(u-\gamma)^{\frac{1}{\gamma}}}{(1-u)(u+\gamma)^{\frac{1}{\gamma}}}\right|^{\operatorname{sign}(\dot{u})}\left|\frac{(1+v)(\gamma-v)^{\frac{1}{\gamma}}}{(1-v)(v+\gamma)^{\frac{1}{\gamma}}}\right|^{\operatorname{sign}(\dot{v})} = d \tag{13}$$

parametrized by $d = \exp\{2\beta_1(1-\gamma^2)\}$ and the signs of $\dot{u}$ and $\dot{v}$.

The trajectory flow

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{\operatorname{sign}(\dot{u})}{\operatorname{sign}\dot{v}}\frac{(1-u^2)(u^2-\gamma^2)}{(1-v^2)(\gamma^2-v^2)}$$

is undefined at the points:

		u	v
v^1	:	1	$-\gamma$
v^2	:	1	γ
focus	:	γ	γ
focus	:	γ	$-\gamma$.

The ground states v^1 and v^2 of the field theory are unstable points of the Hamiltonian flow where an infinite number of trajectories starts at $\tau = -\infty$ and ends at $\tau = +\infty$. The 'time' table for the orbits is also provided by the Hamilton–Jacobi method:

$\frac{\partial J}{\partial i_2} = \beta_2$, where $\beta_2 \in \mathbb{R}$ is another constant, implies

$$-\tau + \operatorname{sign}\dot{u}\int\frac{u^2\,\mathrm{d}u}{\sqrt{\frac{2i_2+2i_1u^2}{u^2-\gamma^2}+(1-u^2)^2(u^2-\gamma^2)}}$$
$$-\operatorname{sign}\dot{v}\int\frac{v^2\,\mathrm{d}v}{\sqrt{\frac{-2i_2-2i_1v^2}{\gamma^2-v^2}+(1-v^2)^2(\gamma^2-v^2)}} = \beta_2.$$

The solution for $i_1 = i_2 = 0$ is

$$\left|\frac{(1+u)(u-\gamma)^{\gamma}}{(1-u)(u+\gamma)^{\gamma}}\right|^{\operatorname{sign}(\dot{u})}\left|\frac{(1+v)(\gamma-v)^{\gamma}}{(1-v)(\gamma+v)^{\gamma}}\right|^{\operatorname{sign}(\dot{v})} = \mathrm{e}^{2(1-\gamma^2)(\beta_2+\tau)}. \tag{14}$$

Numerical analysis, see figure 1, tells us that for regular, finite, values of β_1 and β_2 all the trajectories are homoclinic, i.e. they start and end at the same unstable point $v^{(\pm)}$ with zero momentum. They are infinitely degenerated and intersect at other points, where the flow is undefined, the foci of the ellipse:

$$f_1^2 + \frac{f_2^2}{1-\gamma^2} = 1.$$

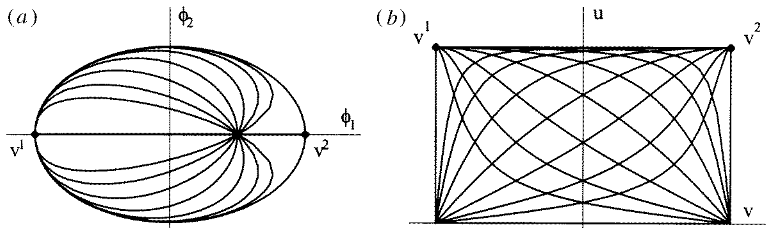

Figure 1. Kink trajectories in the MSTB model. (a) Cartesian plane: paths starting and ending in v^2 which intersect at the focus $(0, \sigma)$ are shown. (b) Elliptic plane: trajectories with both v^1 and v^2 taken as initial/final points are drawn. The focus $(\sigma, -\sigma)$ is the conjugate point to $v^1 = (1, -\sigma)$, while the other focus (σ, σ) and $v^2 = (1, \sigma)$ are conjugate points with respect to each other.

The integration constant β_1 is related to the tangent of the orbit at the focus through which it passes; the velocity of the particle describing a given orbit is determined by β_2.

From the point of view of field theory, there is a nonlinear solitary wave for each trajectory given by (13) and (14).

The topological charge is zero for the whole family and each member is a non-topological kink of two components, both ϕ_1 and ϕ_2 are different from zero:

$$Q_a^{\mathrm{T}}[NTK2(v^1; \beta_1)] = Q_a^{\mathrm{T}}[NTK2(v^2; \beta_1)] = 0.$$

The kink manifold, parametrized by β_1, therefore lives in $\mathcal{C}_0^{++}$ and $\mathcal{C}_0^{--}$ and the translational mode of each solitary wave is given by β_2.

There are also heteroclinic trajectories which arise in the limit $\beta_1 = \infty$; they start from $v^{(\pm)}$ and end at the opposite point, $v^{(\mp)}$. The corresponding kinks and antikinks are both of one and two components and are topological:

$$f^{\mathrm{TK1}}(x) = \begin{pmatrix} \tanh(x - x_0) \\ 0 \end{pmatrix} \qquad f^{\mathrm{AK1}}(x) = \begin{pmatrix} -\tanh(x - x_0) \\ 0 \end{pmatrix}$$
$$f^{\mathrm{NTK2}}(x) = \begin{pmatrix} \tanh \gamma x \\ \bar{\gamma} \operatorname{sech} \gamma x \end{pmatrix} \qquad f^{\mathrm{AK2}}(x) = -\begin{pmatrix} \tanh \gamma x \\ \bar{\gamma} \operatorname{sech} \gamma x \end{pmatrix}$$
$$f^{\mathrm{NTK2}^*}(x) = \begin{pmatrix} \tanh \gamma x \\ -\bar{\gamma} \operatorname{sech} \gamma x \end{pmatrix} \qquad f^{\mathrm{AK2}^*}(x) = -\begin{pmatrix} -\tanh \gamma x \\ \bar{\gamma} \operatorname{sech} \gamma x \end{pmatrix}.$$

They thus live in $\mathcal{C}_1^{+-}$ or $\mathcal{C}_1^{-+}$:

$$Q_1^{\mathrm{T}}(\phi^{\mathrm{NTK2}}) = Q_1^{\mathrm{T}}(\phi^{\mathrm{NTK2}^*}) = Q_1^{\mathrm{T}}(\phi^{\mathrm{TK1}}) = 1$$
$$Q_1^{\mathrm{T}}(\phi^{\mathrm{AK2}}) = Q_1^{\mathrm{T}}(\phi^{\mathrm{AK2}^*}) = Q_1^{\mathrm{T}}(\phi^{\mathrm{AK1}}) = -1.$$

All solitary waves, non-topological and topological kinks and antikinks, come from trajectories that are separatrices between bounded and unbounded motion in phase space. The envelope of the separatrices, itself a separatrix, is formed by the ellipse $\phi_1^2 + \frac{\phi_2^2}{1-\gamma^2} = 1$; the corresponding solitary waves are two-component kinks. Also, the interval inside the ellipse on the real axis $\phi_2 = 0$ is special: it is the steepest descent path and gives the one-component kink.

3. Kink energy sum rules

The energy of any kink is the action of the corresponding trajectory: $E = J_u + J_v$.

$$E = \left[\operatorname{sign} \dot{u} \int \mathrm{d}u\, (1 - u^2) + \operatorname{sign} \dot{v} \int \mathrm{d}v\, (1 - v^2) \right].$$

For all the non-topological kinks, we find a degeneracy in energy:

$$E_{\mathrm{NTK2}} = \frac{\sqrt{2} m^3}{\lambda^2} \left[\frac{2}{3} + \gamma \left(1 - \frac{\gamma^2}{3} \right) \right].$$

The topological kinks have energies,

$$E_{\mathrm{TK1}} = \frac{2}{3} \cdot \frac{\sqrt{2} m^3}{\lambda^2} \qquad E_{\mathrm{TK2}} = \frac{\sqrt{2} m^3}{\lambda^2} \cdot \gamma \left(1 - \frac{\gamma^2}{3} \right).$$

The kink energy sum rule arises:

$$E_{\mathrm{NTK2}} = E_{\mathrm{TK1}} + E_{\mathrm{TK2}}.$$

This means that the trajectory TK1+TK2 is a limiting case of the family NTK2, with an appropiate time rescaling, and leads to the Morse theory interpretation of [14]. A geometric perspective of the sum rule is interesting: we can consider the kink trajectories as geodesics in the Jacobi metric

$$\mathrm{d}s^2 = 2(i_1 + V)\, \mathrm{d}f^a\, \mathrm{d}f^a$$

for the special value $i_1 = 0$. The length of each trajectory

$$l = \int \mathrm{d}\tau \sqrt{g^{ab}(f_1, f_2) \frac{\mathrm{d}f_a}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}f_b}{\mathrm{d}\tau}}$$

in the $u : v$ plane endowed with the Jacobi metric $g^{ab} = 2V\delta^{ab}$ is equal to E, above: $E = l$. Therefore, we see the energy of TK2 as the length of the straight line joining v^1 and v^2 in figure 1 measured with this particular metric. *Simili modo*, the energy of TK1 is the sum of the lengths of the other three straight lines forming the rectangle with corners v^1, v^2 and foci $(\gamma, -\gamma)$ and (γ, γ). The kink energy sum rule means, from this point of view, that all the other curves giving NTK2 kinks have lengths equal to the perimeter of this rectangle. Therefore, there is a visual or graphic procedure to compute kink energies.

3.1. Kink manifold of model A

In this model the action for the mechanical system is:

$$J_{\mathrm{A}} = \int \mathrm{d}\tau \left\{ \frac{1}{2} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}f_a}{\mathrm{d}\tau} + \frac{f_b f_b}{2} (f_a f_a - 1)^2 + \sigma^2 f_2^2 \left(f_a f_a - 1 + \frac{\sigma^2}{2} \right) \right\}$$

and the particle motion equations are:

$$\frac{\mathrm{d}^2 f_c}{\mathrm{d}\tau^2} = f_c (f_a f_a - 1)(3 f_b f_b - 1) + 2\sigma^2 \left(f_c f_2^2 + \delta_{c2} f_2 \left(f_a f_a - 1 + \frac{\sigma^2}{2} \right) \right) \tag{15}$$

to be solved together with the asymptotic conditions:

$$\lim_{\tau \to \pm\infty} f_a(\tau) = v_a^{\mathrm{A}} \qquad \lim_{\tau \to \pm\infty} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} = 0 \qquad v_a^{\mathrm{A}} = \{ v_{\mathrm{A}}^{(\pm 1)},\ v_{\mathrm{A}}^{(\pm i)}, v_{\mathrm{A}}^{(0)} \}. \tag{16}$$

The phase space of the dynamical system is the cotangent bundle T^*M to the configuration space M:

$$M = \mathbb{R}^2 - \{(-\infty, \sigma] \sqcup [-\sigma, \infty)\} \cong \mathbb{R}^+ \times S^1 \sqcup \mathbb{R}^+ \times S^1$$

where $\cong$ means that both sides of the identity are homeomorphic, $\mathbb{R}^+$ is the set of positive real numbers and S^1 is a circle. Three reasons may be invoked to account for this awkward choice of configuration space: (a) trajectories in the real axis, $\phi_2 = 0$, are $e \times \mathbb{Z}_2$ invariant; they are singular in the moduli space of solutions. (b) The Hamiltonian flow is undefined at the points $\phi_2 = 0$, $\phi_1 = \pm\sigma$, the foci of the ellipse $\phi_1^2 + \frac{\phi_2^2}{1-\sigma^2} = 1$. (c) As a consequence of (a) and (b) a half line on the ϕ_1-axis starting at $\phi_1 = \pm\sigma$ must be subtracted. There are two possible choices, depending on the asymptotic behaviour; this amounts topologically to substract from $\mathbb{R}^2$ infinitesimally small disks centred on the foci.

The mechanical Hamiltonian

$$I_1^{\mathrm{A}} = \frac{1}{2} p_a p_a - \frac{1}{2} f_b f_b (f_a f_a - 1)^2 - \sigma^2 f_2^2 \left(f_a f_a - 1 + \frac{\sigma^2}{2} \right)$$

yields the canonical equations

$$\frac{\mathrm{d} f_a}{\mathrm{d}\tau} = \{I_1^{\mathrm{A}}, f_a\} \qquad \frac{\mathrm{d} p_a}{\mathrm{d}\tau} = \{I_1^{\mathrm{A}}, p_a\}.$$

Obviously $\frac{\mathrm{d}I_1^{\mathrm{A}}}{\mathrm{d}\tau} = 0$, but, as in the MSTB model, there is a second invariant

$$I_2^{\mathrm{A}} = \frac{1}{2}\{(f_1 p_2 - f_2 p_1)^2 - \sigma^2 p_2^2\} - \frac{\sigma^2}{2} f_2^2 (\sigma^2 f_1^2 - (f_1^2 + f_2^2 - 1 + \sigma^2)^2) \tag{17}$$

$\frac{\mathrm{d}I_2^{\mathrm{A}}}{\mathrm{d}\tau} = 0$, which is in involution with the Hamiltonian: $\{I_1^{\mathrm{A}}, I_2^{\mathrm{A}}\} = 0$. The dynamical system is completely integrable. The trajectories with the asymptotic behaviour (16) provide all the kinks of the model forming a one-dimensional manifold.

Suitable coordinates in M_{A} are elliptic as in the configuration space of the MSTB model. Notice that the choice of elliptic coordinates enjoys the same pathologies as our configuration space above; in particular, there is an angular coordinate and a real positive one. The 'action' J_{A} is

$$J_{\mathrm{A}} = \int \mathrm{d}\tau \left\{ \frac{1}{2}(u^2 - v^2) \left[\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^2 \right] \right.$$
$$\left. + \frac{1}{2(u^2 - v^2)} [u^2(u^2 - \sigma^2)(1 - u^2)^2 + v^2(\sigma^2 - v^2)(1 - v^2)^2] \right\}$$

in the new variables. The Hamiltonian

$$h^{\mathrm{A}} = \frac{h_u^{\mathrm{A}} + h_v^{\mathrm{A}}}{u^2 - v^2} = [((u^2 - \sigma^2) p_u^2 - u^2(u^2 - \sigma^2)(1 - u^2)^2)$$
$$+ ((\sigma^2 - v^2) p_v^2 - v^2(\sigma^2 - v^2)(1 - v^2)^2)][2(u^2 - v^2)]^{-1}$$

is separable and model A is a Liouville mechanical system of type I, see [12]. In the phase-space submanifold determined by $h^{\mathrm{A}} = i_1^{\mathrm{A}}$, we have $h_u^{\mathrm{A}} - i_1^{\mathrm{A}} = -h_v^{\mathrm{A}} - i_1^{\mathrm{A}} = i_2^{\mathrm{A}}$, where i_2^{A} is another constant. As in the MSTB model, $i_1^{\mathrm{A}} = I_1^{\mathrm{A}}$ while

$$i_2^{\mathrm{A}} = -I_1 - I_2^{\mathrm{A}} \sigma^2.$$

The Hamilton–Jacobi equation

$$\frac{\partial J_{\mathrm{A}}}{\partial \tau} + \mathcal{H}_{\mathrm{A}} \left(\frac{\partial J_{\mathrm{A}}}{\partial u}, \frac{\partial J_{\mathrm{A}}}{\partial v}, u, v \right) = 0$$

is completely separable

$$\begin{aligned}\frac{1}{2}(u^2-\sigma^2)\left(\frac{\mathrm{d}J_u^{\mathrm{A}}}{\mathrm{d}u}\right)-\frac{1}{2}u^2(u^2-\sigma^2)(1-u^2)^2-i_1^{\mathrm{A}}u^2=i_2^{\mathrm{A}}\\ \frac{1}{2}(\sigma^2-v^2)\left(\frac{\mathrm{d}J_v^{\mathrm{A}}}{\mathrm{d}v}\right)-\frac{1}{2}v^2(\sigma^2-v^2)(1-v^2)^2+i_1^{\mathrm{A}}v^2=-i_2^{\mathrm{A}}\end{aligned} \tag{18}$$

if $J_{\mathrm{A}}=J_u^{\mathrm{A}}(u)+J_v^{\mathrm{A}}(v)-i_1^{\mathrm{A}}\tau$. Given the solutions of equation (18) one obtains the equations and the 'time'-table for the trajectories from

$$\frac{\partial J_{\mathrm{A}}}{\partial i_2^{\mathrm{A}}}=\gamma_1^{\mathrm{A}} \qquad \frac{\partial J_{\mathrm{A}}}{\partial i_1^{\mathrm{A}}}=\gamma_2^{\mathrm{A}}$$

where γ_1^{A} and γ_2^{A} are real constants. $J_{\mathrm{A}}<+\infty$ occurs if and only if $i_1^{\mathrm{A}}=i_2^{\mathrm{A}}=0$ and the solitary waves of model A are given by:

$$\mathrm{e}^{2\sigma^2(1-\sigma^2)\gamma_1^{\mathrm{A}}}=\left[\frac{u^2-\sigma^2}{(1-u^2)^{\sigma^2}u^{2(1-\sigma^2)}}\right]^{\mathrm{sign}\,\dot u}\left[\frac{\sigma^2-v^2}{(1-v^2)^{\sigma^2}v^{2(1-\sigma^2)}}\right]^{\mathrm{sign}(\dot v v)}.$$

The model A kink manifold is thus parametrized by γ_1^{A} and the signs of $\dot u$ and $v\dot v$. Also

$$\left(\frac{u^2-\sigma^2}{1-u^2}\right)^{\mathrm{sign}\,\dot u}\left(\frac{\sigma^2-v^2}{1-v^2}\right)^{\mathrm{sign}(v\dot v)}=\mathrm{e}^{2(1-\sigma^2)(\gamma_2^{\mathrm{A}}+\tau)}$$

tells us how u and v depend on τ for the finite action trajectories and/or how u and v depend on x for the kink configurations.

The Hamiltonian flow

$$\frac{\mathrm{d}u}{\mathrm{d}v}=\frac{\mathrm{sign}(\dot u)u(1-u^2)(u^2-\sigma^2)}{\mathrm{sign}(v\dot v)v(1-v^2)(\sigma^2-v^2)}$$

is undefined at the points

		u	v			u	v
$v^1=v_{\mathrm{A}}^{(-1)}$	:	1	$-\sigma$	f_+	:	σ	σ
$v^2=v_{\mathrm{A}}^{(-i)}$	:	1	0	f_-	:	σ	$-\sigma$
$v^3=v_{\mathrm{A}}^{(0)}$	:	σ	0				
$v^4=v_{\mathrm{A}}^{(+i)}$	:	1	0				
$v^5=v_{\mathrm{A}}^{(+1)}$	:	1	σ				
asymptotic points				conjugate points.			

The ground states of the field theory v^1, v^2, v^3, v^4, v^5 are unstable points of the Hamiltonian flow, asymptotically reached by an infinite number of trajectories. We order these points from left to right and from down to up in the Cartesian plane, see figure 2. On the other points where the flow is undefined, the foci $f_\pm$ of the ellipse, an infinite number of trajectories starting and ending at v^2 and v^4 intersect; therefore, f_+ and f_- are conjugate points to v^2 and v^4. We know from numerical analysis that all the trajectories in this model are heteroclinic, see figure 2.

As a novelty with respect to the MSTB model, there are two families that depend on the axis where the asymptotic points are located. This, in turn depends on the relative signs of $\dot v$ and $v\dot v$.

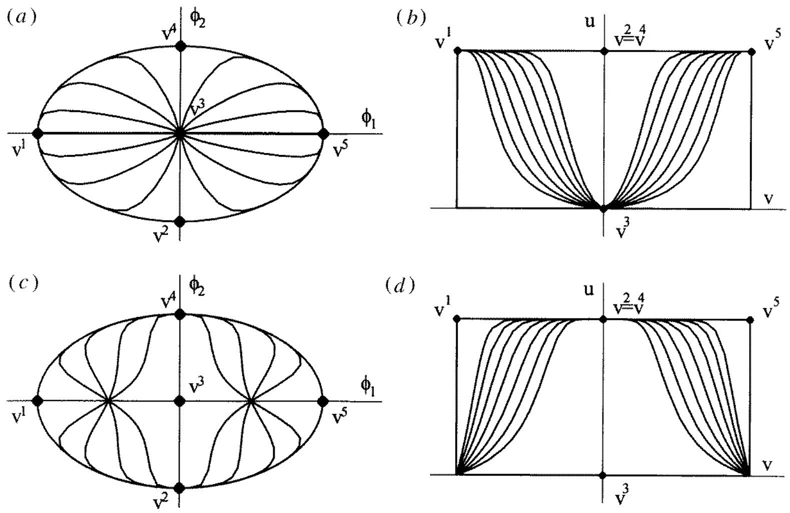

Figure 2. Kink trajectories in model A. (*a*) Trajectories from v^1 and v^3 that terminate at v^3 and v^5 in the Cartesian plane. (*b*) Same trajectories in the elliptic plane. (*c*) Paths starting from v^2 that end at v^4 drawn in the Cartesian plane. Note that the two foci are conjugate points to v^2. (*d*) Same paths in the elliptic plane.

• Family I, plotted in figures 2(*a*) and (*b*). In this case $\text{sign}\,\dot{u} = -\,\text{sign}(v\dot{v})$ and γ_1^A labels each family member according to its tangent at the asymptotic points. γ_2^A provides the 'velocity' of the trajectory and the starting point is determined by $\text{sign}(\dot{u})$.

• Family II, plotted in figures 2(*c*) and (*d*). Now $\text{sign}\,\dot{u} = \text{sign}(v\dot{v})$, and γ_1^A gives the tangent of each trajectory at the foci and γ_2^A chooses the velocity. As above, the starting point is fixed by $\text{sign}(\dot{u})$.

From the point of view of field theory, all these trajectories correspond to solitary waves living in $\mathcal{C}_A^{\alpha\beta}$, where $\alpha\beta$ are: (13), (53), (42) and their opposites (31), (35), (24). All of them are topological kinks of two components or their antikinks. Denoting one such subfamily by $\text{TK2}[\alpha\beta;\gamma_1^A]$, the topological charges are:

• $\text{TK2}[31;\gamma_1^A] \in \mathcal{C}_A^{31}$ and $\text{TK2}[53;\gamma_1^A] \in \mathcal{C}_A^{53}$ have: $Q_1^T = 1$, $Q_2^T = 0$.

For $\text{TK2}[13;\gamma_1^A] \in \mathcal{C}_A^{13}$ and $\text{TK2}[35;\gamma_1^A] \in \mathcal{C}_A^{35}$: $Q_1^T = -1$, $Q_2^T = 0$.

• For $\text{TK2}[24;\gamma_1^A] \in \mathcal{C}_A^{24}$: $Q_2^T = +2$ and $Q_1^T = 0$.

We find $Q_2^T = -2$ and $Q_1^T = 0$ for $\text{TK2}[42;\gamma_1^A]$.

All the topological kinks, both in families I and II, come from trajectories that are separatrices between bounded and unbounded motion in phase space; they arise when $I_1 = I_2 = 0$ a boundary in phase space between unbounded levels sets of functions $I_a = c_a$ and those stratified in invariant tori. The envelope of the separatrices, itself a separatrix, is formed in this model by the ellipse $\phi_1^2 + \frac{\phi_2^2}{1-\sigma^2} = 1$ and the interval on the vertical axis, $\phi_1 = 0$, contained in the domain bounded by the ellipse. Note the difference with the MSTB model where the envelope is only the ellipse.

The analysis suggests that as a trial orbit we should use the curve

$$\phi_1^2 + \frac{\phi_2^2}{1-\sigma^2} = 1 \qquad \text{or} \qquad u = 1 \tag{19}$$

to search for the singular solutions of the Hamilton–Jacobi equations, the limit $\gamma_1^A = \infty$. This affords analytical expressions for special topological kinks of two components in the

Cartesian plane:

$$\phi^{\mathrm{TK2}}(x) = \pm\sqrt{\frac{1}{2}(1+\tanh[\pm\sigma^2(x-x_0)])} \pm \mathrm{i}\sqrt{\frac{(1-\sigma^2)}{2}(1-\tanh[\pm\sigma^2(x-x_0)])} \tag{20}$$

which satify (19) and thus correspond to trajectories on the ellipse. There are eight distinct kinks of this type classified according to the choice of sign in (20), TK2[14] $\in \mathcal{C}_{\mathrm{A}}^{14}$, TK2[12] $\in \mathcal{C}_{\mathrm{A}}^{12}$, TK2[45] $\in \mathcal{C}_{\mathrm{A}}^{45}$, TK2[25] $\in \mathcal{C}_{\mathrm{A}}^{25}$ and the corresponding antikinks. For this, both Q_1^{T} and Q_2^{T} are different from zero. For instance, $Q_1^{\mathrm{T}}(\mathcal{C}_{\mathrm{A}}^{14}) = 1$ and $Q_2^{\mathrm{T}}(\mathcal{C}_{\mathrm{A}}^{14}) = 1$.

Restriction to the imaginary axis immediately gives one-component topological kinks:

$$\phi^{\mathrm{TK1(2)}}(x) = \pm\mathrm{i}\sqrt{\frac{(1-\sigma^2)}{2}(1+\tanh[\pm(x-x_0)]}. \tag{21}$$

There are four kinks of this type, TK1(2)[43] $\in \mathcal{C}_{\mathrm{A}}^{43}$, TK1(2)[32] $\in \mathcal{C}_{\mathrm{A}}^{32}$, plus the two antikinks, for which only Q_2^{T} is non-null.

Finally, we restrict the system to the real axis to find the last type of one-component topological kinks:

$$\phi^{\mathrm{TK1(1)}}(x) = \pm\sqrt{\tfrac{1}{2}(1+\tanh[\pm(x-x_0)]}. \tag{22}$$

For TK1(1)[13] $\in \mathcal{C}_{\mathrm{A}}^{13}$, TK1(1)[35] $\in \mathcal{C}_{\mathrm{A}}^{35}$ and the antikinks $Q_1^{\mathrm{T}} \neq 0$ but $Q_2^{\mathrm{T}} = 0$.

As in the MSTB model, the energy of any kink is the action of the corresponding trajectory: $E_{\mathrm{A}} = J_u^{\mathrm{A}} + J_v^{\mathrm{A}}$.

$$E_{\mathrm{A}} = \mathrm{sign}(\dot{u}) \int u(1-u^2)\,\mathrm{d}u + \mathrm{sign}(v\dot{v}) \int v(1-v^2)\,\mathrm{d}v.$$

We obtain:

$$E_{\mathrm{A}}(\mathrm{TK1(1)}) = \frac{\sqrt{2}m^3}{4\lambda^2} \qquad E_{\mathrm{A}}(\mathrm{TK1(2)}) = \left(\frac{1}{4} - \frac{\sigma^2}{2}\left(1-\frac{\sigma^2}{2}\right)\right)\frac{\sqrt{2}m^3}{\lambda^2}$$

$$E_{\mathrm{A}}(\mathrm{TK2}) = \frac{\sigma^2}{2}\left(1-\frac{\sigma^2}{4}\right)\frac{\sqrt{2}m^3}{\lambda^2} \qquad E_{\mathrm{A}}(\mathrm{TK2}[13;\gamma_1^{\mathrm{A}}])) = \frac{\sqrt{2}m^3}{4\lambda^2}$$

$$E_{\mathrm{A}}(\mathrm{TK2}[24;\gamma_1^{\mathrm{A}}])) = \frac{\sqrt{2}m^3}{2\lambda^2}$$

and therefore the kink energy sum rules, designating by TK2$_I$, the kinks in $\mathcal{C}_{\mathrm{A}}^{13}$ $\mathcal{C}_{\mathrm{A}}^{34}$, and by TK2$_{II}$ those in $\mathcal{C}_{\mathrm{A}}^{42}$,

$$E_{\mathrm{A}}(\mathrm{TK1(1)}) = E_{\mathrm{A}}(\mathrm{TK2}) + E_{\mathrm{A}}(\mathrm{TK1(2)}) = E_{\mathrm{A}}(\mathrm{TK2}_I[\gamma_1^{\mathrm{A}}]) \tag{23}$$

$$E_{\mathrm{A}}(\mathrm{TK2}_{II}[\gamma_1^{\mathrm{A}}]) = 2E_{\mathrm{A}}(\mathrm{TK2}_I[\gamma_1^{\mathrm{A}}]) = E_{\mathrm{A}}(\mathrm{TK1(2)}) + E_{\mathrm{A}}(\mathrm{TK1(1)}) + E_{\mathrm{A}}(\mathrm{TK2}). \tag{24}$$

Again, these kink energies and sum rules can be read directly from the diagrams in figure 2 in the elliptic plane, simply by measuring the length of the trajectories with respect to the Jacobi metric, which is the same for *every* curve with equal starting and ending points.

The sum rules for the kink energies contain a great deal of information. TK1(1) and TK2 + TK1(2) appear at the singular limits, $\gamma_1^{\mathrm{A}} \rightarrow \pm\infty$ and can be included in family I because of the identity in the energies (23). TK1(1) is special because it exhibits more symmetry than the others members in TK2$_I[\gamma_1^{\mathrm{A}}]$. From (24) we interpret TK1(2) + TK1(1) + TK2 as arising in the limits $\gamma_1^{\mathrm{A}} = \pm\infty$ of the family TK2$_{II}[\gamma_1^{\mathrm{A}}]$.

3.2. Kink manifold of model B

The action for the associated dynamical model is:

$$J_{\mathrm{B}} = \int \mathrm{d}\tau \, \left\{ \frac{1}{2} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}f_a}{\mathrm{d}\tau} + (4f_1^2 + f_2^2 - 1)^2 + 4f_1^2 f_2^2 \right\}$$

and the Euler–Lagrange equations are

$$\frac{\mathrm{d}^2 f_1}{\mathrm{d}\tau^2} = 4f_1[4(4f_1^2 + f_2^2 - 1) + 2f_2^2] \qquad \frac{\mathrm{d}^2 f_2}{\mathrm{d}\tau^2} = 4f_2[(4f_1^2 + f_2^2 - 1) + 2f_1^2] \tag{25}$$

and finite action trajectories show the asymptotic behaviour:

$$\lim_{\tau\mapsto\pm\infty} f_a(\tau) = v_a^{\mathrm{B}}(\pm\infty) \qquad \lim_{\tau\mapsto\pm\infty} \frac{\mathrm{d}f_a}{\mathrm{d}\tau} = 0 \qquad v^{\mathrm{B}} = \{v_{\mathrm{B}}^{(\pm 1)},\ v_{\mathrm{B}}^{(\pm i)}\}. \tag{26}$$

In this model, the configuration space M_{B} is:

$$M_{\mathrm{B}} = \mathbb{R}^2 - \{(-\infty, 0] \sqcup [0, +\infty)\} = \mathbb{R}^+ \times \mathbb{R}$$

for similar reasons to those explained in model A. In this case however, the flow is undefined at the origin $\phi_1 = \phi_2 = 0$. Therefore, parabolic coordinates $u \in (-\infty, \infty)$, $v \in [0, \infty)$ such that

$$\phi_1 = \tfrac{1}{2}(u^2 - v^2) \qquad \phi_2 = uv$$

should make model B more tractable. Note that the origin is the common focus of the parabolas

$$\phi_2^2 = 1 \pm 2\phi_1 \qquad \text{or} \qquad v = 1 \qquad \text{and} \qquad u = \pm 1.$$

Besides the Hamiltonian

$$I_1^{\mathrm{B}} = \tfrac{1}{2} p_a p_a - (4f_1^2 + f_2^2 - 1)^2 - 4f_1^2 f_2^2$$

generating the ‘time’ evolution,

$$\dot{f_a} = \{I_1^{\mathrm{B}}, f_a\} \qquad \dot{p_a} = \{I_1^{\mathrm{B}}, p_a\} \qquad \dot{I}_1^{\mathrm{B}} = 0$$

there is a second invariant, $\dot{I}_2^{\mathrm{B}} = 0$,

$$I_2^{\mathrm{B}} = (f_1 p_2 - f_2 p_1) p_2 + 4f_1 f_2^2 (2f_1^2 + f_2^2 - 1)$$

in involution with I_1^{B}, $\{I_1^{\mathrm{B}}, I_2^{\mathrm{B}}\} = 0$. Therefore, the dynamical system is completely integrable. Observe that the first term in I_2^{B} is a certain projection of the angular over the linear momentum: $w = -j^{12}(\boldsymbol{p} \wedge \boldsymbol{e}_1)\boldsymbol{e}_2$.

In the field theory framework, we have quadratic functions of the time- and/or space-derivatives of the fields,

$$L_2 = \left(\phi_1 \frac{\partial \phi_2}{\partial x_0} - \phi_2 \frac{\partial \phi_1}{\partial x_0}\right) \cdot \frac{\partial \phi_2}{\partial x_0} + 4\phi_1 \phi_2^2 (2\phi_1^2 + \phi_2^2 - 1)$$

$$I_2 = \left(\phi_1 \frac{\partial \phi_2}{\partial x_1} - \phi_2 \frac{\partial \phi_1}{\partial x_1}\right) \cdot \frac{\partial \phi_2}{\partial x_1} + 4\phi_1 \phi_2^2 (2\phi_1^2 + \phi_2^2 - 1)$$

with some nonlinear dependence of the form:

$$F\left(\frac{\partial L_2}{\partial x_0}, \frac{\partial I_2}{\partial x_0}\right) = 0.$$

Unlike the MSTB model, there is no limit where this identity becomes a linear continuity equation. We have not written a similar relation in model A because the situation is identical to that of the MSTB model.

The action J_B in parabolic coordinates is:

$$J_B = \int d\tau \left\{ \frac{1}{2}(u^2+v^2)\left[\left(\frac{du}{d\tau}\right)^2 + \left(\frac{dv}{d\tau}\right)^2\right] + \frac{1}{u^2+v^2}[u^2(u^4-1)^2 + v^2(v^4-1)^2]\right\}.$$

The Hamiltonian in the new variables reads

$$h_B = \frac{h_u + h_v}{u^2+v^2} = \frac{[p_u^2 - 2u^2(u^4-1)^2] + [p_v^2 - 2v^2(v^4-1)^2]}{2(u^2+v^2)}$$

where $p_u = (u^2+v^2)\dot{u}$ and $p_v = (u^2+v^2)\dot{v}$.

The dynamical system associated with model B is a Liouville mechanical system of type III according to the classification in [12]. In the phase space submanifold characterized by $h_B = i_1^B$, we have $h_u - i_1^B u^2 = -h_v + i_1^B v^2 = i_2^B$. Obviously, $i_1^B = I_1^B$ while

$$I_2^B = -i_2^B.$$

We now understand why the MSTB and A models depend on a continuous parameter and there is no such freedom in model B; this is due to the characteristics of the coordinate system in which the models are separable. Once the origin on the plane is fixed, a foliation by confocal ellipses and hyperbolas requires the choice of foci. The common focus of the parabolas in a parabolic system can be chosen, however, as the origin.

The Hamilton–Jacobi equation

$$\frac{\partial J_B}{\partial \tau} + \mathcal{H}_B\left(\frac{\partial J_B}{\partial u}, \frac{\partial J_B}{\partial v}, u, v\right) = 0$$

becomes completely separable by writing $J_B = J_u^B(u) + J_v^B(v) + i_1^B \tau$:

$$\begin{aligned} \frac{1}{2}\left(\frac{dJ_1^B}{du}\right)^2 - u^2(u^4-1)^2 - i_1^B u^2 &= i_2^B \\ \frac{1}{2}\left(\frac{dJ_2^B}{dv}\right)^2 - v^2(v^4-1)^2 - i_1^B v^2 &= -i_2^B. \end{aligned} \tag{27}$$

From the solutions of (27), the equations and the 'time'-table for the trajectories can be immediately written:

$$\frac{\partial J_B}{\partial i_2^B} = \gamma_1^B \qquad \frac{\partial J_B}{\partial i_1^B} = \gamma_2^B \tag{28}$$

where γ_1^B, γ_2^B are undetermined real constants. The finite action $W_B < +\infty$ requires $0 = i_1^B = i_2^B$ in (28) and the kinks of model B correspond to solutions of the equations:

$$\left(\frac{u^4}{1-u^4}\right)^{\operatorname{sign}(u\dot{u})}\left(\frac{1-v^4}{v^4}\right)^{\operatorname{sign}\dot{v}} = e^{4\sqrt{2}\gamma_1^B}$$

so that the kink manifold of model B is parametrized by γ_1^B and the sign of $u\dot{u}$. The 'time' evolution of each trajectory is scheduled according to the equation:

$$\left(\frac{1+u^4}{1-u^4}\right)^{\operatorname{sign}(u\dot{u})}\left(\frac{1+v^4}{1-v^4}\right)^{\operatorname{sign}(\dot{v})} = e^{4\sqrt{2}(\gamma_2^B+\tau)}.$$

The Hamiltonian flow

$$\frac{dv}{du} = \frac{\operatorname{sign}(\dot{v})v|1-v^4|}{\operatorname{sign}(u\dot{u})u|1-u^4|}$$

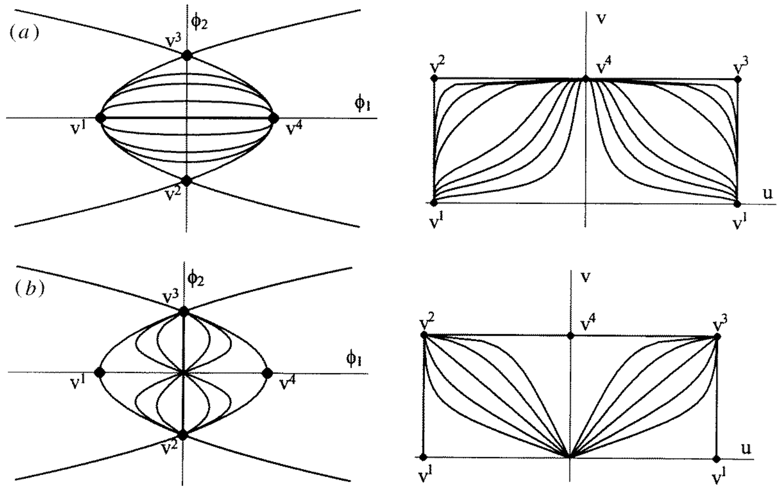

Figure 3. (*a*) Stable kink trajectories in model B. (*b*) Unstable kink trajectories in model B.

is undefined at the points

		v	u			v	u
$v^4 = v_{\mathrm{B}}^{(+1)}$	:	1	0	f	:	0	0
$v^3 = v_{\mathrm{B}}^{(+i)}$	:	1	1				
$v^2 = v_{\mathrm{B}}^{(-i)}$	:	1	−1				
$v^1 = v_{\mathrm{B}}^{(+1)}$	:	0	1				

asymptotic points conjugate point.

Again, the ground states v^1, v^2, v^3, v^4 of the field theory are unstable points of the Hamiltonian flow ordered from left to right and from down to up in the Cartesian plane, see figure 3: an infinite number of trajectories of finite action J_{B} asymptotically reach those points. The focus of the parabolas $\phi_2^2 = 1 \pm 2\phi_1$, the origin, is crossed by infinite paths with endpoints in v^3 and v^4 and thus f is a conjugate point to v^2 and v^4. As in model A, numerical analysis shows that all the trajectories are heteroclinic, see figure 3. There are also two families of orbits depending on the relative signs of $u\dot{u}$ and $\dot{v}$.

• Family I, plotted in figure 3(*a*). Here $\mathrm{sign}(u\dot{u}) = -\,\mathrm{sign}(\dot{v})$. The parameters γ_1^{B} and γ_2^{B} play exactly the same role as their cousins γ_1^{A} and γ_2^{A} in family I of model A.

• Family II, plotted in figure 3(*b*). Now $\mathrm{sign}(u\dot{u}) = \mathrm{sign}(\dot{v})$ and the situation is parallel to that for family II in model A.

In $(1+1)$-dimensional field theory all these trajectories give rise to solitary ways. Bearing in mind the same notation as in model A, we have the families of topological kinks of two components.

• $\mathrm{TK2}[14, \gamma_1^{\mathrm{B}}] \in \mathcal{C}_{\mathrm{B}}^{14}$, $\mathrm{TK2}[41, \gamma_1^{\mathrm{B}}] \in \mathcal{C}_{\mathrm{B}}^{41}$ with topological charges $Q_1^{\mathrm{T}} = \pm 1$ and $Q_2^{\mathrm{T}} = 0$.

• $\mathrm{TK2}[23, \gamma_1^{\mathrm{B}}] \in \mathcal{C}_{\mathrm{B}}^{23}$, $\mathrm{TK2}[32, \gamma_1^{\mathrm{B}}] \in \mathcal{C}_{\mathrm{B}}^{32}$ and topological charges $Q_2^{\mathrm{T}}(\mathcal{C}_{\mathrm{B}}^{23}) = 2 = -Q_2^{\mathrm{T}}(\mathcal{C}_{\mathrm{B}}^{32})$, $Q_1^{\mathrm{T}}(\mathcal{C}_{\mathrm{B}}^{23}) = Q_1^{\mathrm{T}}(\mathcal{C}_{\mathrm{B}}^{32}) = 0$.

All the topological kinks come from separatrices in the dynamical system. The envelope of the separatrices is given by the portion of the parabola $\phi_2^2 = 1 - 2\phi_1$ contained in the domain bounded by $\phi_2^2 = 1 + 2\phi_1$ plus the portion of the parabola $\phi_2^2 = 1 + 2\phi_1$ starting and ending at the curve $\phi_2^2 = 1 - 2\phi_1$.

We use the parabolas

$$\phi_2^2 = 1 \pm 2\phi_1 \qquad \text{or} \qquad v = 1 \qquad \text{and} \qquad u = 1$$

as trial orbits to find the singular solutions of the Hamilton–Jacobi equations at the limit $\gamma_1^{\mathrm{B}} = \infty$. We find analytic expressions for two-component topological kinks living on the parabolas:

$$\phi^{\mathrm{TK2}}(x) = \pm\tfrac{1}{4}(1 - \tanh[\pm 2\sqrt{2}(x - x_0)]) \pm \mathrm{i}\sqrt{\tfrac{1}{2}(1 + \tanh[\pm 2\sqrt{2}(x - x_0)])}.$$

There are eight distinct kinks of this type:

TK2[13] $\in \mathcal{C}_{\mathrm{B}}^{13}$, TK2[34] $\in \mathcal{C}_{\mathrm{B}}^{34}$, TK2[12] $\in \mathcal{C}_{\mathrm{B}}^{12}$, TK2[24] $\in \mathcal{C}_{\mathrm{B}}^{24}$ plus the four antikinks. For this, both Q_1^{T} and Q_2^{T} are different from zero. For instance, $Q_1^{\mathrm{T}}(\mathcal{C}_{\mathrm{B}}^{13}) = 1$, $Q_2^{\mathrm{T}}(\mathcal{C}_{\mathrm{B}}^{13}) = 1$.

One-component topological kinks also exist and are obtained by either restricting ourselves to the imaginary or the real axis:

$$\phi^{\mathrm{TK1(2)}}(x) = \mathrm{i}\tanh[\pm\sqrt{2}(x - x_0)]$$
$$\phi^{\mathrm{TK1(1)}}(x) = \tfrac{1}{2}\tanh[\pm 2\sqrt{2}(x - x_0)].$$

We have TK1(2)[32] $\in \mathcal{C}_{\mathrm{B}}^{32}$ plus the antikink and TK1(1)[14] $\in \mathcal{C}_{\mathrm{B}}^{14}$ and its antikink. Note that in model B the ϕ_a-axis is not an envelope of the separatrices.

The energy of any kink is the action of the corresponding trajectory: $E_{\mathrm{B}} = J_1^{\mathrm{B}} + J_2^{\mathrm{B}}$.

$$E_{\mathrm{B}} = \sqrt{2}\,\mathrm{sign}(u\dot{u}) \int u(u^4 - 1)\,\mathrm{d}u + \sqrt{2}\,\mathrm{sign}(\dot{v}) \int v(v^4 - 1)\,\mathrm{d}v.$$

We find

$$E_{\mathrm{B}}(\mathrm{TK2}) = \frac{2}{3}\frac{m^3}{\lambda^2} \qquad E_{\mathrm{B}}(\mathrm{TK1(1)}) = E_{\mathrm{B}}(\mathrm{TK2}[14; \gamma_1^{\mathrm{B}}]) = \frac{4}{3}\frac{m^3}{\lambda^2}$$
$$E_{\mathrm{B}}(\mathrm{TK2}[23; \gamma_1^{\mathrm{B}}]) = \frac{8}{3}\frac{m^3}{\lambda^2} \qquad E_{\mathrm{B}}(\mathrm{TK1(2)}) = \frac{8}{3}\frac{m^3}{\lambda^2}$$

and the following kink energy sum rules:

$$2E_{\mathrm{B}}(\mathrm{TK2}) = E_{\mathrm{B}}(\mathrm{TK2}[14; \gamma_1^{\mathrm{B}}]) = E_{\mathrm{B}}(\mathrm{TK1(1)})$$
$$E_{\mathrm{B}}(\mathrm{TK2}[23; \gamma_1^{\mathrm{B}}]) = E_{\mathrm{B}}(\mathrm{TK1(2)}) = 2E_{\mathrm{B}}(\mathrm{TK2}[14; \gamma_1^{\mathrm{B}}]).$$

The kink energy sum rules can immediately be visualized in figures 3(a) and (b) because the energies are exactly given by the lengths of the corresponding trajectories with respect to the corresponding Jacobi metric. There are also singular limits that we summarize as follows.

In family I, the limits $\gamma_1^{\mathrm{B}} \to \pm\infty$ include TK1(1) as well as either TK2[13] + TK2[34] or TK2[12] + TK2[24], because the first kink energy sum rule.

In family II, the analogous limits correspond to either TK2[24] + TK1(1)[41] + TK2[13] or TK2[21] + TK1(1)[14] + TK2[43], according to the other sum rule.

4. Further comments

We have not dealt with the important issue of kink stability in this paper. In general, the solitary waves are not stable if the corresponding trajectories cross any conjugate point. In quantum theory there are *bona fide* quantum states built on the classical kinks only if they are stable. In that case it is possible to compute the mass and the wave functionals of the quantum kinks along the lines put forward in [14] for the MSTB solitons. The analysis was based on study of Morse Theory *à la* Bott for the configuration space of the MSTB model. Moreover, the Witten–Smale version of Morse theory as implemented in [16] allowed us to compute the decay amplitudes of the non-stable kinks in the MSTB. It seems plausible that similar ideas would work for the kinks of models A and B with small variations and we plan to develop the Morse theory of models A and B in future research.

In order to compute the quantum kink mass of conventional kinks, supersymmetry helps because it sometimes supresses quantum corrections and the classical result is exact. This suggests that the supersymmetric extension of models A and B, if possible, would permit easy computation of the properties of the quantum kinks, enlarging the manifold of supersymmetric models with important topological content.

Finally, we understand that models A and B are by no means unique. First, it is not difficult to obtain field models with associated mechanical problems which are dynamical systems of Liouville type II and IV, i.e. separable by using either Cartesian or polar coordinates. In a similar vein, there are models of type A and B generalizing models A and B, i.e. with associated mechanical systems separable respectively in elliptic and parabolic coordinates, with a richer structure of kink manifolds.

References

[1] Jackiw R and Schrieffer R 1981 *Nucl. Phys.* B **190** 253
[2] Scott A, Chu F and McLaughlin D 1973 *Proc. IEEE* **61** 1443
[3] Kibble T W 1980 *Phys. Rep.* **67** 183
Vilenkin A 1985 *Phys. Rep.* **121** 1
[4] Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North-Holland)
[5] Mermin D 1979 *Rev. Mod. Phys.* **51** 591
[6] Montonen C 1976 *Nucl. Phys.* B **112** 349
Sarker S, Trullinger S W and Bishop A R 1976 *Phys. Lett.* **59A** 255
[7] Coleman S 1973 *Commun. Math. Phys.* **31** 259
[8] Magyari E and Thomas H 1984 *Phys. Lett.* **100A** 11
[9] Hieratinta J 1983 *Phys. Lett.* **96A** 273
Hieratinta J 1987 *Phys. Rep.* **147** 87
Fordy A P 1983 *Phys. Lett.* **97A** 21
[10] Ito H 1985 *Phys. Lett.* **112A** 119
[11] Jackiw R and Weinberg E 1990 *Phys. Rev. Lett.* **64** 2334
[12] Liouville J 1849 *J. Math. Phys. Appl.* **11** 345
Perelomov A M *Integrable Systems of Classical Mechanics and Lie Algebras* (Boston, MA: Birhäuser)
[13] Zamolodchikov A 1989 *Adv. Stud. Pure Math.* **19** 642
[14] Mateos Guilarte J 1987 *Lett. Math. Phys.* **14** 169
Mateos Guilarte J 1988 *Ann. Phys.* **188** 307
[15] Olive D, Turok N and Underwood J 1993 *Nucl. Phys.* B **401** 663
[16] Mateos Guilarte J 1992 *Ann. Phys.* **216** 122